Supplementary Material for the paper "A Bayesian Robust Kalman Smoothing Framework for State-Space Models with Uncertain Noise Statistics"

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In this supplement, we provide more details on the likelihood function $f(\mathcal{Y}_L|\boldsymbol{\theta})$ needed for obtaining the posterior effective noise statistics based on the Metropolis Hastings technique.

Algorithm 1 summarizes the procedure for computing the likelihood function of the parameter $\boldsymbol{\theta} = [\theta_1, \theta_2]$ given a sequence of observations $\mathcal{Y}_L = \{\mathbf{y}_0, ...\mathbf{y}_L\}$ up to time L. The inputs to this algorithm are the sequence of observations $\mathcal{Y}_L$, the initial conditions $\mathrm{E}[\mathbf{x}_0]$ and $\mathrm{cov}[\mathbf{x}_0]$, and matrices $\mathbf{\Phi}_k$, $\mathbf{\Gamma}_k$, $\mathbf{H}_k$, $\mathbf{Q}^{\theta_1}$, and $\mathbf{R}^{\theta_2}$ characterizing the parameterized state-space model in (1) and (2).

In order to compute the likelihood function in Line 17, which is based on equation (39) in the paper, we first need to obtain S_L and matrices $\mathbf{\Sigma}_L$ and $\mathbf{M}_L$ using recursive calculations outlined in lines 8 to 14. After the recursive calculations, $\mathbf{\Delta}_L$ and $\mathbf{G}_L$ can be computed using $\mathbf{\Sigma}_L$ and $\mathbf{M}_L$.

In each iteration of the recursive calculations, first $\mathbf{\Sigma}_{k+1}$ is obtained using $\mathbf{\Lambda}_k$. Then $\mathbf{\Sigma}_{k+1}$ along with $\mathbf{\Lambda}_k$, $\mathbf{\Sigma}_k$, and $\mathbf{M}_k$ is used to calculate $\mathbf{M}_{k+1}$. In the next step, $\mathbf{\Sigma}_{k+1}$ and $\mathbf{M}_{k+1}$ can be used to update the value of S_k to S_{k+1}. Also, $\mathbf{\Lambda}_{k+1}$ is found using $\mathbf{\Sigma}_{k+1}$. Finally, $\mathbf{W}_{k+1}$ can be obtained via $\mathbf{\Sigma}_{k+1}$ and $\mathbf{M}_{k+1}$.

Algorithm 1 Likelihood Function Computation

input: $\mathcal{Y}_L = \{\mathbf{y}_0, \mathbf{y}_1, ..., \mathbf{y}_L\}$, $\mathrm{E}[\mathbf{x}_0]$, $\mathrm{cov}[\mathbf{x}_0]$, $\mathbf{\Phi}_k$, $\mathbf{\Gamma}_k$, $\mathbf{H}_k$, $\mathbf{Q}^{\theta_1}$, $\mathbf{R}^{\theta_2}$

output: $f(\mathcal{Y}_L|\theta)$

$S_0 \leftarrow 1$ ▷ the initialization step for S_k

$\mathbf{\Sigma}_0 \leftarrow \mathrm{cov}[\mathbf{x}_0]$ ▷ the initialization step for $\mathbf{\Sigma}_k$

$\mathbf{M}_0 \leftarrow \mathrm{E}[\mathbf{x}_0]$ ▷ the initialization step for $\mathbf{M}_k$

$\widetilde{\mathbf{Q}}_0^{\theta_1} \leftarrow \mathbf{\Gamma}_0 \mathbf{Q}^{\theta_1} \mathbf{\Gamma}_0^T$

$\mathbf{\Lambda}_0^{-1} \leftarrow \mathbf{\Phi}_0^T (\widetilde{\mathbf{Q}}_0^{\theta_1})^{-1} \mathbf{\Phi}_0 + \mathbf{H}_0 (\mathbf{R}^{\theta_2})^{-1} \mathbf{H}_0 + \mathbf{\Sigma}_0^{-1}$ ▷ the initialization step for $\mathbf{\Lambda}_k$

for $k = 0 : L-1$ **do**

$\quad \widetilde{\mathbf{Q}}_k^{\theta_1} \leftarrow \mathbf{\Gamma}_k \mathbf{Q}^{\theta_1} \mathbf{\Gamma}_k^T$

$\quad \mathbf{\Sigma}_{k+1}^{-1} \leftarrow (\widetilde{\mathbf{Q}}_k^{\theta_1})^{-1} - (\widetilde{\mathbf{Q}}_k^{\theta_1})^{-1} \mathbf{\Phi}_k \mathbf{\Lambda}_k \mathbf{\Phi}_k^T (\widetilde{\mathbf{Q}}_k^{\theta_1})^{-1}$ ▷ equation (42)

$\quad \mathbf{M}_{k+1} \leftarrow \mathbf{\Sigma}_{k+1} (\widetilde{\mathbf{Q}}_k^{\theta_1})^{-1} \mathbf{\Phi}_k \mathbf{\Lambda}_k \Big(\mathbf{H}_k^T (\mathbf{R}^{\theta_2})^{-1} \mathbf{y}_k + \mathbf{\Sigma}_k^{-1} \mathbf{M}_k \Big)$ ▷ equation (43)

$\quad S_{k+1} \leftarrow S_k \sqrt{\frac{|\mathbf{\Lambda}_k| \, |\mathbf{\Sigma}_{k+1}|}{|\widetilde{\mathbf{Q}}_k^{\theta_1}| \, |\mathbf{\Sigma}_k|}} \mathcal{N}\big(\mathbf{y}_k; \mathbf{0}_{m\times 1}, \mathbf{R}^{\theta_2}\big) \exp\left(\frac{\mathbf{M}_{k+1}^T \mathbf{\Sigma}_{k+1}^{-1} \mathbf{M}_{k+1} + \mathbf{W}_k^T \mathbf{\Lambda}_k \mathbf{W}_k - \mathbf{M}_k^T \mathbf{\Sigma}_k^{-1} \mathbf{M}_k}{2} \right)$
▷ equation (44)

$\quad \mathbf{\Lambda}_{k+1}^{-1} \leftarrow \mathbf{\Phi}_{k+1}^T (\widetilde{\mathbf{Q}}_{k+1}^{\theta_1})^{-1} \mathbf{\Phi}_{k+1} + \mathbf{H}_{k+1} (\mathbf{R}^{\theta_2})^{-1} \mathbf{H}_{k+1} + \mathbf{\Sigma}_{k+1}^{-1}$ ▷ equation (45)

$\quad \mathbf{W}_{k+1} \leftarrow \mathbf{H}_{k+1}^T (\mathbf{R}^{\theta_2})^{-1} \mathbf{y}_{k+1} + \mathbf{\Sigma}_{k+1}^{-1} \mathbf{M}_{k+1}$ ▷ equation (46)

$\mathbf{\Delta}_L^{-1} \leftarrow \mathbf{H}_L^T (\mathbf{R}^{\theta_2})^{-1} \mathbf{H}_L + \mathbf{\Sigma}_L^{-1}$ ▷ equation (40)

$\mathbf{G}_L \leftarrow \mathbf{\Delta}_L \Big(\mathbf{H}_L^T (\mathbf{R}^{\theta_2})^{-1} \mathbf{y}_L + \mathbf{\Sigma}_L^{-1} \mathbf{M}_L \Big)$ ▷ equation(41)

$f(\mathcal{Y}_L|\boldsymbol{\theta}) \leftarrow S_L \sqrt{\frac{|\mathbf{\Delta}_L|}{|\mathbf{\Sigma}_L|}} \mathcal{N}\big(\mathbf{y}_L; \mathbf{0}_{m\times 1}, \mathbf{R}^{\theta_2}\big) \exp\Big(\frac{1}{2} \big(\mathbf{G}_L^T \mathbf{\Delta}_L^{-1} \mathbf{G}_L - \mathbf{M}_L^T \mathbf{\Sigma}_L^{-1} \mathbf{M}_L \big) \Big)$ ▷
equation (39)

return $f(\mathcal{Y}_L|\boldsymbol{\theta})$